Summary of Fundamental Limitations in Feedback Design (LTI SISO Systems)

From Chapter 6 of
A FIRST GRADUATE COURSE IN FEEDBACK CONTROL
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Limitation	Conditions	Description	Effects
Time Domain Design Limitations			
Interpolation Constraints	(a) Feedback system in Figure-1 is stable (b) Plant and/or Controller have CRHP poles/zeros	Let the plant and controller be described as:- $P(s) = \frac{N_P(s)}{D_P(s)} \qquad C(s) = \frac{N_C(s)}{D_C(s)}$ $where\ (N_P(s), D_P(s))\ and\ (N_C(s), D_C(s))\ are\ each\ pairs\ of\ coprime\ polynomials$ (a) Suppose that $P(z) = 0$ and/or $C(z) = 0, z \in CRHP$. Then necessarily $S(z) = 1 \qquad T(z) = 0$ (b) Suppose $p \in CRHP$ is a pole of $P(s)$ and/or $C(s)$. Then necessarily $S(p) = 0 \qquad T(p) = 1$	If the plant has CRHP zeros or poles, then the values of closed loop transfer functions $S(s)$ and $T(s)$ are constrained to equal certain values at the locations of these zeros and poles. So, such plants may be difficult to control, with the degree of difficulty depending on the relative location of these poles and zeros with respect to the control bandwidth. The above are true for zeros/poles in Plant/Controller. We cannot control poles/zeros of Plant but we can control those of the Controller
Integrators and Overshoot	(a) Feedback system in Figure-1 is stable (b) Unit step command: $r(t) = 1(t)$ (c) L(s) has integrators	(a) If $L(s)$ has at least two poles at $s = 0$:- $\int_0^\infty e(t)dt = 0 \qquad [e(t) = r(t) - y(t)]$ (b) If $L(s)$ has one pole at $s = 0$:- $\int_0^\infty e(t)dt = \frac{1}{K_v} \qquad [K_v \triangleq \lim_{s \to 0} sL(s)]$	In practice $L(s)$ will be strictly proper, and the initial value theorem states that the step response will start at zero. So, there will be a time interval for which $e(t) > 0$. The relations on the left say that there must be a time interval for which $e(t) < 0$ which means that the response must exhibit an overshoot.

ORHP Poles and Overshoot Relation of Rise Time and Overshoot	(a) Feedback system in Figure-1 is stable (b) Unit step command: $r(t) = 1(t)$ (c) L(s) has ORHP poles	Suppose that p is a pole of $L(s)$ with $Re(p) > 0$. Then $$\int_0^\infty e^{-pt} e(t)\, dt = 0$$ Define α rise time t_α as the smallest value of t such that $$y(t) \leq \alpha < 1 \quad \forall t \leq t_\alpha$$ Then $$pt_\alpha \leq \log\left(\frac{1 + y_{OS} - \alpha}{1 - \alpha}\right)$$	If $Re(p) > 0$ then the system will necessarily exhibit overshoot. Indeed, there exists a minimum overshoot whose size is inversely proportional to a measure of the rise time of the system. As p becomes more unstable, the rise time of the system must decrease to maintain the desired overshoot.
ORHP Zeros and Undershoot Relation of Settling Time and Undershoot	(a) Feedback system in Figure-1 is stable (b) Unit step command: $r(t) = 1(t)$ (c) L(s) has ORHP zeros *(Nonminimum phase zeros)*	Suppose that z is a zero of $L(s)$ with $Re(z) > 0$. Then $$\int_0^\infty e^{-zt} y(t) dt = 0$$ Define β settling time,t_β as the smallest value of t such that $$\lvert y(t) - 1 \rvert \leq \beta < 1 \qquad \forall t \geq t_\beta$$ Then $$zt_\beta \geq \log\left(\frac{y_{US} - 1 + \beta}{y_{US}}\right)$$	If $Re(z) > 0$ the system must exhibit undershoot (except when $L(s)$ is identically zero, in which case there will be no step response) As z gets smaller, the settling time must increase in order to maintain the desired level of undershoot
Effect ORHP Poles and Zeros (both) on Overshoot and Undershoot	(a) Feedback system in Figure-1 is stable (b) Unit step command: $r(t) = 1(t)$ (c) L(s) has a real ORHP pole and a real ORHP zero (d) $T(0) = 1$	$L(s)$ has a real ORHP pole p and a real ORHP zero z. Then $If\ p > z, then$ $$y_{US} < \frac{z}{z - p}$$ $If\ z > p, then$ $$y_{OS} > \frac{p}{z - p}$$	If L(s) has both a real ORHP pole and a real ORHP zero, the undershoot will be less than a certain amount (depending on the locations of the pole and zero), and the overshoot will be greater than a certain amount (depending on the locations of the pole and zero)

Frequency Domain Design Specifications			
Frequency Dependent Bounds on SISO Transfer functions	---	The bounds take the form $$\lvert S(j\omega)\rvert < M_S(\omega) \quad (Bound\ 1)$$ $$\lvert T(j\omega)\rvert < M_T(\omega) \quad (Bound\ 2)$$ $$\lvert C(j\omega)S(j\omega)\rvert < M_{CS}(\omega) \quad (Bound\ 3)$$ $$\lvert S(j\omega)P(j\omega)\rvert < M_{SP}(\omega) \quad (Bound\ 4)$$	In fact, such specifications may not be stated explicitly but are implicitly present in the feedback design problem. Frequency response design specifications are sometimes stated in terms of infinity norm. So, we examine the Bode gain plots of closed loop transfer functions for a peak that is "too large" or bandwidth that is "too high"
Algebraic Design Tradeoffs	---	$$S(s) + T(s) = 1$$ $$M_S(\omega) + M_T(\omega) > 1 \quad (if\ Bound\ 1\ and\ 2\ are\ satisfied)$$ $$M_S(\omega) + \lvert P(j\omega)\rvert M_{CS}(j\omega) > 1 \quad (if\ Bound\ 1\ and\ 3\ are\ satisfied)$$ $$M_T(\omega) + \frac{M_{SP}(j\omega)}{\lvert P(j\omega)\rvert} > 1 \quad (if\ Bound\ 2\ and\ 4\ are\ satisfied)$$ $$\frac{M_{SP}(j\omega)}{\lvert P(j\omega)\rvert} + \lvert P(j\omega)\rvert M_{CS}(j\omega) > 1 \quad (if\ Bound\ 3\ and\ 4\ are\ satisfied)$$	The first one is a fundamental design tradeoff that does not let $S(s)$ and $T(s)$ to be high at the same frequency. Rest of the four can be proved using the first bound, and triangular inequality.
Bode Gain-Phase Relation For Rational Transfer functions with no poles and zeros in ORHP	(a) $L(s) = P(s)C(s)$ is a rational function with no poles or zeros in ORHP (b) Gain has been normalized so that $L(0^+) > 0$	Then at each frequency ω_0, the phase and gain are related by $$\angle L(j\omega_0) = \frac{2\omega_0}{\pi}\int_{-\infty}^{+\infty} \frac{\log\lvert L(j\omega)\rvert - \log\lvert L(j\omega_0)\rvert}{\omega^2 - \omega_0^2}\, d\omega$$ $$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{d\log\lvert L(j\omega_0)e^{v}a\rvert}{dv}\log\left\{\coth\frac{\lvert v\rvert}{2}\right\} dv \qquad \left[v = \log\left(\frac{\omega}{\omega_0}\right)\right]$$	The phase of the transfer function is completely determined by its gain. Hence, we have only one degree of freedom in design. The phase depends upon the slope of the Bode gain plot. Rate of gain decrease near cross over frequency cannot be much greater than $-20\ dB/decade$, if we have to achieve nominal stability with reasonable phase margin.

Effect of RHP zeros and poles	(a) $L(s)$ has N_Z ORHP zeros $\{z_i . i = 1, \dots, N_Z\}$ and N_P ORHP poles $\{p_i . i = 1, \dots, N_p\}$	Then we may factor $L(s)$ as $L(s) = L_0(s) B_z(s) B_p(s)^{-1}$ where $L_0(s)$ does not have any ORHP poles or zeros (Bode gain-phase relation is applicable to $L(s)$) $$B_z(s) = \prod_{i=1}^{N_Z} \frac{z_i - s}{\bar{z}_i + s} \;, B_p(s) = \prod_{i=1}^{N_P} \frac{p_i - s}{\bar{p}_i + s}$$ are the Blaschke products of ORHP zeros and poles respectively $\lvert B_z(j\omega) \rvert = \lvert B_p(j\omega) \rvert = 1 \,, \forall \omega$ The gain of each Blaschke product remains 1 for all frequencies, but the phase changes B_z adds a phase lag without changing gain, and B_p adds a phase lead without changing gain	The ORHP (NMP) contribute additional phase lag, thereby worsening the design tradeoff between high gain at low frequencies and low gain at high frequencies. The presence of NMP zero implies that the gain cross over frequency must lie well below $\omega = z$, where the NMP zero contributes $-90\ deg$ phase lag. The ORHP poles contribute a phase lead, which does not mitigate the design tradeoff.
Time Delays	L(s) is of the form $L(s) = L_0(s) e^{-s\tau}$	Time delay adds phase lag to $L(j\omega)$ without affecting $\lvert L(j\omega) \rvert$	Time delay can be approximated as a rational transfer function by using Pade approximations
Bode Sensitivity Integral (the tradeoff described by Bode gain-phase relation, in terms of sensitivity function)	(a) $L(s)$ is factorized as:- $L_0(s) B_z(s) B_p(s)^{-1} e^{-s\tau}$ (τ represents a time delay) (b) $S(s)$ is stable (c) $L(s)$ has at least two more poles than zeros (d) $L(s)$ has no poles in ORHP	Then $$\int_0^\infty \log(\lvert S(j\omega) \rvert) = 0$$ The integral relation states that if $\lvert S(j\omega) \rvert < 1$ over some frequency interval, then necessarily $\lvert S(j\omega) \rvert > 1$ at other frequencies. The area of sensitivity increase equals the area of sensitivity decrease in units of $dB \ \times \frac{rad}{s}$	This is also called Bode waterbed effect, because the area of sensitivity increase may be obtained by allowing $\log\lvert S(j\omega) \rvert$ to exceed zero by an arbitrarily small amount ϵ over an arbitrarily large frequency range. This tail of the sensitivity looks like a waterbed. But, this small increase cannot be maintained over a large frequency range, as explained next.
Effect of Bandwidth Limitations	(a) $\lvert T(j\omega) \rvert < \epsilon \left(\frac{\omega_c}{\omega} \right)^k$ $\forall\ \omega > \omega_c$ $\epsilon < \frac{1}{2} and\ k \geq 2$	Then $$\int_{\omega_c}^\infty \log\lvert S(j\omega) \rvert \, d\omega < \frac{3\epsilon\omega_c}{2(k-1)}$$	This shows that the area under the peak of the sensitivity function is bounded, it follows that most of the tradeoff between sensitivity reduction

	(can also assume that L satisfies this bound)		and sensitivity peak must occur at low and intermediate frequencies
Lower bound on a Peak in Sensitivity	(a) $L(s)$ is factorized as:- $L_0(s)B_z(s)B_p(s)^{-1}e^{-s\tau}$ (τ represents a time delay) (b) $S(s)$ is stable (c) $L(s)$ has at least two more poles than zeros (d) $L(s)$ has no poles in ORHP (e) $\lvert S(j\omega)\rvert \le \alpha < 1$ $\forall\ \omega \le \omega_1$ (f) $\lvert T(j\omega)\rvert < \epsilon\left(\frac{\omega_c}{\omega}\right)^k$ $\forall\ \omega > \omega_c$ $\epsilon < \frac{1}{2} and\ k \ge 2$ $\omega_c > \omega_1$	Then $$\max_{\omega\in(\omega_1,\omega_c)} \log\lvert S(j\omega)\rvert \ge \left(\frac{\omega_1}{\omega_c-\omega_1}\right)\log\frac{1}{\alpha} - \left(\frac{\omega_c}{\omega_c-\omega_1}\right)\frac{3\epsilon}{2(k-1)}$$	This provides a lower bound on a peak in sensitivity at intermediate frequencies. A peak will necessarily occur. *Sensitivity peak reduces phase margin.* It follows that bandwidth constraints, which are always present in a realistic control design will impose a non trivial tradeoff between sensitivity reduction at low frequencies and sensitivity increase at high frequencies.
Sensitivity for an Open Loop Unstable System	(a) $S(s)$ is stable (b) $L(s)$ has at least two more poles than zeros (c) $L(s)$ has ORHP poles $\{p_i . i = 1, \dots, N_p\}$	Then $$\int_0^\infty \log\lvert S(j\omega)\rvert\, d\omega = \pi\sum_{i=1}^{N_p} Re(p_i)$$	The tradeoff imposed by Bode sensitivity integral worsens when there are open loop poles in ORHP. It follows that the area of sensitivity increase exceeds that of sensitivity increase by an amount proportional to the distances from the ORHP poles to the left half plane. The cost of achieving two benefits of feedback, disturbance attenuation and stabilization, is thus greater than that of achieving only benefit of disturbance attenuation.
Poisson Sensitivity Integral	(a) $S(s)$ is stable	Let $z = x + jy$ $(x > 0)$ denote an NMP zero of $L(s)$. Then	We know that the additional phase lag due to NMP zero may cause poor feedback properties

		$$\int_0^\infty \log\lvert S(j\omega)\rvert\, W(z,\omega)d\omega = \pi \log\lvert B_p^{-1}(z)\rvert$$ where $$W(z,\omega) = \frac{x}{x^2+(y-\omega)^2} + \frac{x}{x^2+(y+\omega)^2}$$ If the system is open loop stable, the RHS is greater that zero, indeed $$\log\lvert B_p^{-1}(s)\rvert = \sum_{i=1}^{N_p} \log\left\lvert\frac{\bar{p}_i+z}{p_i-z}\right\rvert$$	such as phase margins, if the lag is significant over the useful bandwidth of the system. $\lvert S(j\omega)\rvert$ will be less than 1 at certain frequencies, and greater than 1 at other frequencies. But, the areas are weighted, and need not be equal. *There exists a guaranteed peak in $\lvert S(j\omega)\rvert$ even without the assumption of additional bandwidth constraint, due to unstable pole.* This follows from the weighting function in the integrand, which implies that the weighted length of the $j\omega$-axis us finite. In particular, systems with approximate pole zero cancellations have very poor sensitivity and robustness properties.
Sensitivity Peak and Phase Lag		Suppose that $z = x > 0$ is a real NMP zero. Assume sensitivity function is required to be small at low frequencies $$\lvert S(j\omega)\rvert < \alpha < 1 \quad \forall\omega \in \Omega \triangleq [0,\omega_0) \qquad \boxed{1}$$ Define the weighted length of the frequency interval by $$\Theta(z,\Omega) \triangleq \int_0^{\omega_0} W(z,\omega)\, d\omega$$ It turns out that the weighted length of the interval Ω is equal to minus the additional phase lag contributed by the NMP zero at its upper endpoint: $$\Theta(z,\Omega) = -\angle\left(\frac{z-j\omega_0}{z+j\omega_0}\right)$$ In particular $\Theta(z,\Omega) \to \pi$ as $\omega_0 \to \infty$ Now, if the closed loop system is stable and $\boxed{1}$ is satisfied, then	There is a lower bound on the peak of $\lvert S(j\omega)\rvert$ at $\omega \geq \omega_0$. Before ω_0, $\lvert S(j\omega)\rvert < \alpha$ and after ω_0 the peak is defined by the last relation on the left. It can be said that NMP zeros lying well outside the bandwidth do not impose severe performance tradeoffs. Essentially, the loop gain $\lvert L(j\omega)\rvert$ must be rolled off well below the frequency at which the phase lag contributed by the zero becomes significant.

		$$\max_{\omega\geq\omega_0}\lvert S(j\omega)\rvert > \left(\frac{1}{\alpha}\right)^{\frac{\Theta(z,\Omega)}{\pi-\Theta(z,\Omega)}}\left\lvert B_p^{-1}(z)\right\rvert^{\frac{\pi}{\pi-\Theta(z,\Omega)}}$$	
Open Loop System has both a NMP zero and an ORHP Pole	$\lvert L(s)\rvert$ has both an NMP zero and an ORHP pole	$$\lVert S\rVert_\infty \geq \left\lvert B_p^{-1}(z)\right\rvert$$	If the open loop system has both an NMP zero and an ORHP pole, then there will exist a peak in sensitivity even if no specification such as $\lvert S(j\omega)\rvert < \alpha < 1$ is imposed
Middleton Complimentary sensitivity Integral (Dual to Bode Sensitivity Integral)	(a) $\lvert T(s)\rvert$ is stable (b) $\lvert L(s)\rvert$ has at least one integrator	Then $$\int_0^\infty \log\lvert T(j\omega)\rvert\frac{d\omega}{\omega^2}+\frac{\pi}{2K_v}=\pi\sum_{i=1}^{N_z}\frac{1}{z_i}+\frac{\pi\tau}{2}$$ $where\ K_v\ is\ the\ velocity\ constant\ [K_v \triangleq \lim_{s\to 0} sL(s)]\ and\ \tau\ is\ time\ delay$	The Poisson sensitivity integral holds for each NMP zero, taken one at a time, but does not incorporate the combined effects of multiple zeros. The size of the summation on RHS is an indication of design difficult associated with NMP zeros. If the summation is large, then $\lvert T(j\omega)\rvert$ will tend to have a peak at intermediate frequencies whose size is influenced by other design specifications. This peak will make $\lvert T(j\omega)\rvert$ large at intermediate frequencies, whereas for better noise performance, we want $\lvert T(j\omega)\rvert$ to be small at intermediate frequencies.
Poisson Complimentary Sensitivity Integral (Dual to Poisson Sensitivity Integral)	(a) $\lvert T(s)\rvert$ is stable (b) $\lvert L(s)\rvert$ has an ORHP pole	Suppose that $p = x + jy, x > 0$ is an ORHP pole of $L(s)$ $$\int_0^\infty \log\lvert T(j\omega)\rvert\, W(p,\omega)d\omega = \pi\log\lvert B_z^{-1}(p)\rvert + \pi\tau x$$	It follows that any open loop unstable system which also has an NMP zero or a time delay must have a peak in complementary sensitivity that satisfies the lower bound $$\lVert T\rVert_\infty \geq \lvert B_z^{-1}(p)\rvert e^{\tau x}$$

		where $B_z(s)$ is the Blaschke product of ORHP zeros of $L(s)$ and $$W(p,\omega) = \frac{x}{x^2 + (y-\omega)^2} + \frac{x}{x^2 + (y+\omega)^2}$$	Time delays will contribute to design limitations associated with an open loop unstable system.

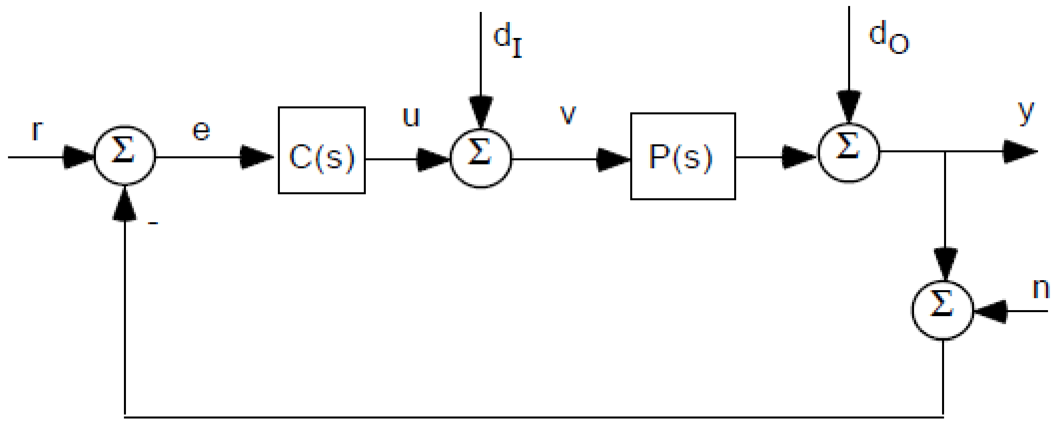

Figure-1: One degree of freedom SISO feedback system